Investigating the behavior of a function modeling the depth of a towfish

Rationale

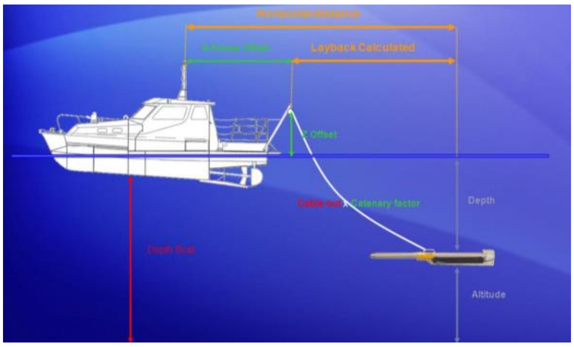

Figure 1 Schematic showing a towfish vessel system (Townsend)

This investigation aims to investigate the behavior of a function that models the depth of a towfish traveling at a constant velocity. As shown in figure 1 towfish are missile-shaped probes towed at specific depths behind boats to study the seafloor at depths out of reach from active sonar on the surface (Wikmark). Towfish are a significant resource for subsea industry, military endeavors, and scientific investigation (Wikmark). Despite the applicability of towfish there has been little development in their operational methods in the last 20 years (Wikmark). Currently, towfish manage their depth by varying the length of rope they are tethered to, and micro-adjustments are done by use of fins and engines, essentially depth is mechanically varied by direct application of force to the towfish (Wikmark). This blunt approach may be outdated and inefficient, and there is great room for development. Recently there has been great developments in the aerospace industry with companies like Boeing revolutionizing airplane design with the new X-37B spaceplane (Wall). The significance of the X37-B is that it generates lift by using the aerodynamics of its body rather than wings, which has proven to be an efficient space shuttle design (Wall). Perhaps the concept of using a body's inherent aerodynamic properties to its advantage can be applied to towfish design as well. The equivalent to the X-37B in towfish design would be designing a towfish that varies its depth by changing its environment rather than applying a brute force directly to itself, for example increasing the drag experienced by a towfish would cause its depth to decrease. To apply this new method of towfish depth management, a mathematical model of how varying the environmental conditions a towfish is under, changes its depth must be explored. This technology could potentially save companies, and researchers millions and is thus essential to be investigated. This exploration is however only concerned with the basics of how a towfish responds to changes in its environment, more specifically its only concerned with mathematically modeling how the towfish behaves when at a constant velocity. But the investigation strongly urges other researchers to build on the concepts and ideas presented in this investigation.

In this investigation a function that models the depth of a towfish based on the behavior of the vector forces acting on the towfish was investigated. The success of the function was then evaluated by investigating the behavior of the function when its argument was set as the towfish's velocity and its geometry. The investigation produced the following function, whose behavior was analyzed and shown to be effective when modeling the effect of varying the towfish's velocity under ideal conditions:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{\frac{1}{2}\rho C_D A \vec{v}^2}{\sqrt{\left(\frac{1}{2}\rho C_D A \vec{v}^2\right)^2 + \left(\vec{g}(\Delta\rho i V_n + \rho V - m)\right)^2}} \right)^2_i}$$

The function f models the depth of the towfish as any one of the parameters is varied (under ideal conditions). Due to the expanse of this investigation, it was split into several components which were explored individually these 3 sections of the exploration are:

1. Conjecturing a function that models the behavior of a towfish tethered to a straight wire (a very simplified model)

2. Conjecturing a function that models the behavior of a towfish tethered to a wire that changes shape in response to its environment (a more realistic model)
3. Exploring the behavior of the function

Introduction to the task

When a towfish is towed by a towing vessel, as shown in figure 2, there are a variety of vector forces acting on the system. If the vector sum of all forces acting on the system is 0 at any given time, the system will have a defined shape (the towfish will have a constant position), but if the vector sum is greater or less than 0, the system will move until it reaches a state where the vector sum of all forces are 0 (equilibrium)(Allum and Talbot). Thus it's essential to understand the behavior of the forces acting on the system to conjecture a function modeling the depth of the towfish. The forces shown in figure 2 are explained in greater detail below.

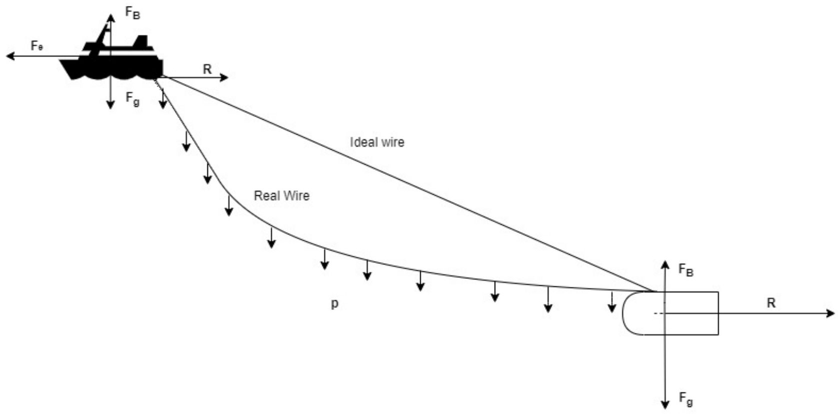

Figure 2 free body diagram showing all the forces acting on the system. where vectors are presented in bold

The vector forces acting on the towfish at any time are the force of gravity, buoyancy, and drag. These vector forces act on the towfish as well as the rope connecting the towfish to the towing vessel (the vector forces acting on the towfish are shown in figure 2). Gravitational force is the fundamental force of attraction between any 2 masses in the universe (Allum and Talbot), it can be calculated on earth using Newtons second law (Allum and Talbot):

$$\overrightarrow{F_g} = -m\vec{g}$$

Where $\overrightarrow{F_g}$ is the gravitational force acting on the object, m is the objects mass, and $\vec{g}$ is the acceleration due to gravity (Allum and Talbot).

The buoyant force is an upwards force that always acts in opposition to the force of gravity (Allum and Talbot). The buoyant force is caused by a difference in pressure between the bottom of a vessel and the top when submerged in a fluid (Allum and Talbot). Mathematically the buoyant force can be expressed through Archimedes principle which states that the buoyant force acting on an object when submerged is equal to the weight of the fluid displaced, which is summarized in the below relation:

$$\overrightarrow{F_B} = \rho\vec{g}V$$

Where $\overrightarrow{F_B}$ is the buoyant force acting on the object, ρ is the density of the fluid, $\vec{g}$ is the acceleration due to gravity, and V is the volume of the displaced fluid (Allum and Talbot). The vectors $\overrightarrow{F_g}$ and $\overrightarrow{F_B}$ are shown in figure 2 to only act on the towfish and towing vessel, although they act on the wire as well. The ideal wire is neutrally buoyant so the vector sum of $\overrightarrow{F_g}$ and $\overrightarrow{F_B}$ is 0, thus no vectors are shown on the ideal wire. While the real wire, is negatively buoyant which is shown by the force in the figure denoted by $\vec{p}$. $\vec{p}$ is the distributed net force of gravity, which is the vector sum of $\overrightarrow{F_g}$ and $\overrightarrow{F_B}$ acting on every point in the wire (Allum and Talbot). $\vec{p}$ is the force that causes the curved shape of the real wire which is known as a catenary curve (Svirin).

The third force that acts on the towfish is the force of drag. Drag is a frictional force caused by collisions between particles of the fluid and the object in motion, during each of these collisions the object transfers some of its kinetic energy to the particles of the fluid, thus drag always acts opposite to the direction of motion (Allum and Talbot). Mathematically drag can be expressed using the drag equation:

$$\vec{R} = \frac{1}{2}\rho C_D A \vec{v}^2$$

Where $\vec{R}$ is drag, $\vec{v}$ is the velocity, ρ is the density of the fluid, C_D is the drag coefficient of the object, and A is the reference area, which is the cross-sectional area of the object on the plane perpendicular to the direction of motion (Allum and Talbot).

The vector forces acting on the towing vessel are what causes the towfish to move. Since the investigation is only concerned with the behavior of the towfish when at a constant velocity the vector sum of the force $\overrightarrow{F_e}$ (engine output) and $\vec{R}$ drag can always be assumed to be 0 which causes the towfish to be at a constant velocity.

All these forces can be expressed in terms of the force of internal tension in the ideal or the real wire. Thus the behavior of the function for the depth of the towfish will be mathematically treated as wholly dependent on the forces presented in the diagram in terms of internal tension.

Towfish dynamics

The simplest possible model of a towfish system that to some extent reflects reality is a towfish tethered to an ideal wire (Wikmark). An ideal wire is a neutrally buoyant wire that behaves like a rigid body, meaning that the application of a force to the body doesn't change its shape (Wikmark). The first model considered in this investigation was the ideal wire model as it's the simplest model that reflects reality, and can thus serve as a good starting point for further exploration.

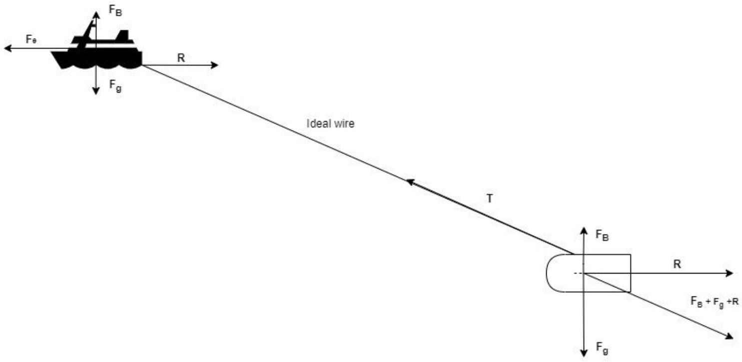

Figure 3 Free body diagram showing vector forces acting on the idealised system

To conjecture a function modeling the behavior of the idealized system the following 2 axioms were considered along with figure 3:

1. For the towfish to have a defined constant position the vector sum of all forces acting on the system must be 0
2. In an ideal wire, the internal tension vector must be colinear with the wire.

The first axiom is valid at all times because if the vector sum of forces acting on the towfish is greater than or less than 0 the unbalanced force will cause the system to move until it gets to a position where the vector sum of forces is 0. The significance of this axiom is that at all times the vector sum of $\vec{T}$, $\overrightarrow{R(v)}$, $\overrightarrow{F_g}$, and $\overrightarrow{F_B}$ must be 0. Thus:

$$-\vec{T} = \vec{R} + \overrightarrow{F_g} + \overrightarrow{F_B}$$

This property of the tension vector is significant when considered along with the second axiom; since the tension vector and the ideal wire are always colinear; the direction of the vector $\vec{T}$ determines the angle the wire will take to the x-axis, and thus determines the depth of the towfish, as shown in figure 4. Since $\vec{R}$ only has a non-zero x component, and $\overrightarrow{F_B}$ and $\overrightarrow{F_g}$ only have non-zero y components the tension vector $\vec{T}$ colinear with the wire is:

$$\vec{T} = -\begin{pmatrix} \vec{R} \\ \vec{F_B} + \vec{F_g} \end{pmatrix}$$

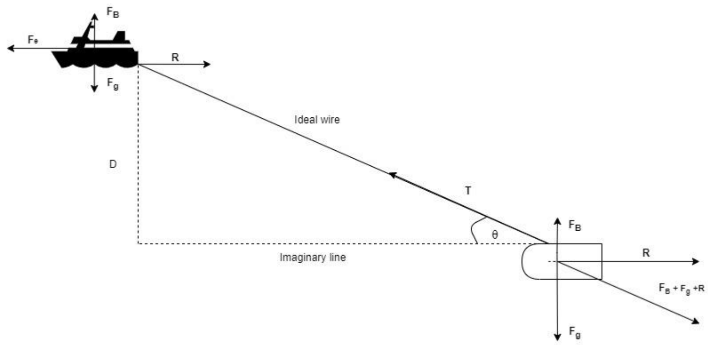

Figure 4 Sketch showing the theoretical triangle formed between the wire and the axes

Since the vector $\vec{T}$ determines the angle the ideal wire takes to the x axis, by applying Pythagoras theorem the depth of the towfish can be given as follows by considering the wire as the hypotenuse of a right angled triangle as shown in figure 4:

$$D = l \sin \theta$$

Where D is the depth of the towfish, l is the length of the wire, and θ is the angle between the tension vector and the x axis. To make this function applicable to the sensory data available to towfish vessels a function for θ was explored.

The angle between the tension vector and the x axis can be expressed in 2 ways by considering the x and y components of $\vec{T}$ in cartesian form:

$$\vec{T} = -\begin{pmatrix} \overrightarrow{R(v)} \\ \vec{F_B} + \vec{F_g} \end{pmatrix} = \begin{cases} x = |\vec{T}| \cos \theta = -\vec{R} \\ y = |\vec{T}| \sin \theta = -(\vec{F_B} + \vec{F_g}) \end{cases}$$

Thus the function for the angle between the tension vector and the x axis that uses relevant sensory data is:

$$\theta = sin^{-1}\left(\frac{-(\vec{F_B} + \vec{F_g})}{|\vec{T}|}\right)$$

Or:

$$\theta = cos^{-1}\left(\frac{-\vec{R}}{|\vec{T}|}\right)$$

Thus the function that models the depth of a towfish tethered to an ideal wire is:

$$D = lsin\left(sin^{-1}\left(\frac{-(\vec{F_B} + \vec{F_g})}{|\vec{T}|}\right)\right)$$

Which can be simplified to:

$$D = l\frac{-(\vec{F_B} + \vec{F_g})}{|\vec{T}|}$$

Which in expanded form is:

$$D = l\frac{-\vec{g}(V\rho - m)}{\sqrt{\left(\frac{1}{2}\rho C_D A\vec{v}^2\right)^2 + (\vec{g}(V\rho - m))^2}}$$

Where D is the depth of the towfish, l is the length of the wire, $\vec{g}$ is gravitational acceleration, V is the volume of the towfish, ρ is the density of water, C_D is the drag coefficient of the towfish, A is the cross sectional area of the towfish, $\vec{v}$ is the velocity of the towfish, and m is the mass of the towfish.

The function $D(\vec{v})$ is shown plotted in figure 5 with parameters set as realistic values (the exact values chosen are unimportant):

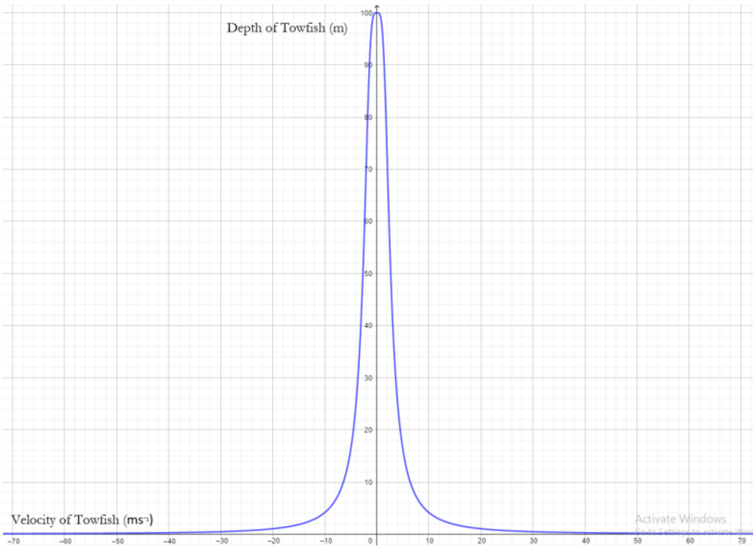

Figure 5 Plot of function D(v) showing the depth of the towfish plotted against its velocity

Figure 5 shows the function D(v) plotted on a cartesian co-ordinate system. Figure 5 shows that the function D(v) takes a bell shape that rapidly converges to y=0. Figure 5 supports the accuracy of the function, because the wire length was set to 100 and figure 5 shows that the maximum depth that can be reached by the towfish is 100 m. It also shows that the function D(v) is very responsive to changes in v, which makes sense, as velocity in the drag equation is squared.

Although the accuracy of this function is qualitatively supported by figure 5 its still only able to model the depth of the towfish under ideal conditions where the flow velocity of the fluid is constant, and the wire pulling the towfish is considered as an ideal non rigid body. Thus, the function likely estimates how a function modeling the depth of a towfish would behave, however it doesn't actually model how the depth of a towfish changes in response to its environment. Thus this exploration was continued to explore how a function modeling a towfish tethered to a non rigid wire behaves.

<u>Towfish with a real wire dynamics</u>

The next logical question to ask is how the behavior of the function changes when the towfish is tethered to a real wire as shown in figure 6. A real wire is a wire that behaves like a non-rigid body, which are bodies that change shape in response to the vector forces acting on them (Wikmark). To conjecture a function that models the depth of a towfish attached to a real wire the following axioms of the wires behavior must be considered:

1. For a non-rigid body to have a defined constant shape the vector sum of all forces acting on it must be 0
2. Tension vectors are always tangential to the point in a wire they act on

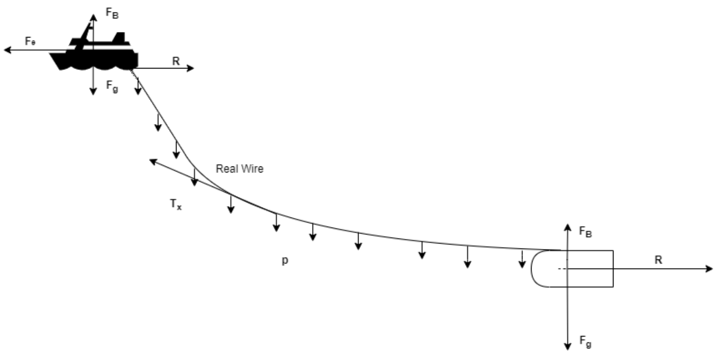

Figure 6 free body diagram showing the vectors acting on the towfish system when the towfish is tethered to a real wire

The first axiom can be summarized in terms of internal tension; all the forces acting on a point in the wire can be expressed in terms of one tension vector, and the vector sum of all the tension vectors on all the points in the wire must equal 0 for the wire to have a defined constant shape. The second axiom is of special importance, it essentially says that the gradient of a tension vector at point x is the same as the gradient of the line at point x. Thus when the sum of the internal tension vectors in the wire is 0 the following function f(x) should model the shape of the wire:

$$f(x) = \int \overrightarrow{T_x}\, dx + c$$

Where $\overrightarrow{T_x}$ is the tension vector tangential to the wire at point x. due to axiom 1 stated above, $\overrightarrow{T_x}$ is equal and opposite to all the vector forces that act on any point below point x:

$$\overrightarrow{T_x} = -\begin{pmatrix} \vec{R} + \overrightarrow{R_x} \\ \vec{p} + \overrightarrow{F_B} + \overrightarrow{F_g} \end{pmatrix}$$

Where $\vec{R}$ is the drag acting on the towfish, and $\overrightarrow{R_x}$[1] is the drag acting on a small segment of wire at which point x is on (drag cannot act on a point), and $\vec{p}$ is the distributed force of gravity and buoyancy acting on any point >x, $\overrightarrow{F_B}$ is the buoyant force acting on the towfish, and $\overrightarrow{F_g}$ is the force of gravity acting on the towfish. Ideally this vector would be investigated to conjecture a function f(x) modelling the depth of a towfish in response to various parameters, however this is impossible without advanced computing software, because the x component of the vector contains a loop dependency. The function f(x) depends on $\overrightarrow{T_x}$ which depends on $\overrightarrow{R_x}$, but $\overrightarrow{R_x}$ depends on the cross sectional area of wire exposed to the fluid thus $\overrightarrow{R_x}$ is dependent on f(x) and vice versa, thus an insolvable loop occurs. This loop is however eliminated if the wire is considered to be very thin, making $\overrightarrow{R_x}$ is negligible so the tension vector on any point in the thin wire is:

$$\overrightarrow{T_x} = -\begin{pmatrix} \vec{R} \\ \vec{p} + \overrightarrow{F_B} + \overrightarrow{F_g} \end{pmatrix}$$

The vector $\overrightarrow{T_x}$ can be used to conjecture a function f(x), but since vector calculus is out of the scope of this exploration a workaround had to be explored, which was done by considering the conditions allowing $\overrightarrow{T_x}$ to model the wire. As mentioned in the first axiom of the wires behavior, a non-rigid body will only have a constant shape when the vector sum of all forces acting on it is 0. Thus, the only condition that allows the function f(x) to model the depth of the towfish is when:

$$\overrightarrow{T_x} = \overrightarrow{T_{x+\Delta x}}$$

For any values of x and x+Δx along the wire. By considering the vector $\overrightarrow{T_x}$ and figure 6 it becomes clear that the only component of $\overrightarrow{T_x}$ that changes between a point x and x+Δx is $\vec{p}$. This is because $\vec{p}$ is the only vector that is directly intersecting the wire as seen in figure 6. Since the only parameter that changes between points in the wire is $\vec{p}$ and $\vec{p}$ is a parameter of the y component of $\overrightarrow{T_x}$ the conditions that allow f(x) to model the depth of the towfish can be specified to:

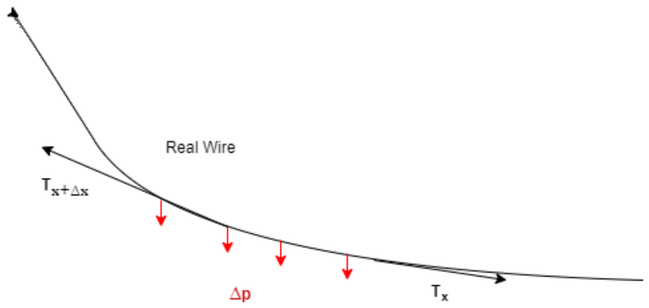

Figure 7 Free body diagram showing how $\vec{p}$ changes between 2 points in the wire

$$\left|\overrightarrow{T_x}\right| \sin\alpha + \left|\overrightarrow{\Delta p}\right| = \left|\overrightarrow{T_{x+\Delta x}}\right| sin\beta$$

For any values of x and x+Δx along the wire. Where α and β are the angles between the vectors and the x axis, and $\overrightarrow{\Delta p}$ is the difference in $\vec{p}$ between points x and x+Δx, this important property is displayed clearly in figure 7.

Since the only parameter that changes between 2 points in the wire is $\vec{p}$, the rate of change of $\vec{p}$ thus directly determines the rate of change of f(x) and its behavior was thus further investigated to conjecture f(x).

$\vec{p}$ is the distributed force of gravity acting on any point on the curve and is given by:

[1] The vector $\overrightarrow{R_x}$ is not shown on figure 6 because its neglected as explained later in the paragraph

$$\vec{p} = (\rho_{water} - \rho_{wire})(V\vec{g})$$

Where ρ is density, V is the volume of the wire below the given point in the wire, and $\vec{g}$ is gravitational acceleration. The function above shows that $\vec{p}$ is a multiple of $\vec{g}$ so the x component of $\vec{p}$ will always be 0 just like $\vec{g}$, so the function for $\vec{p}$ can be considered a scalar function. When treating the function for $\vec{p}$ as scalar function it takes the form of a straight line function (y=mx):

$$|\vec{p}|(V) = |\vec{g}|(\rho_{wire} - \rho_{warter})(V)$$

Where V is the argument of the function and $|\vec{g}|(\rho_{warter} - \rho_{wire})$ is the gradient of the function. Thus, the rate of change of V directly determines the rate of change of $\vec{p}$ which determines the rate of change of f(x). To use the volume of wire below any specific point in the wire to conjecture f(x) the volume of wire below point x would have to be expressed in terms of x, which is impossible since volume of wire below point x determines f(x) but f(x) determines the volume of wire below x. This makes the function f(x) an insolvable loop in its current state.

Thus the function f(x) cannot be expressed as:

$$f(x) = \int |\overrightarrow{T_x}|\, dx + c$$

However the function f(x) can be approximated by splitting the wire into n segments of length l that are assumed to be straight. Then by calculating the height of each straight segment using basic trigonometric techniques, and summing up the height of every segment of the wire an approximation of f(x) is created (an approximation of the depth of the towfish). Since each segment has defined dimensions, the volume of wire below any segment number c can be calculated as the product of the segment number and the volume per segment.

Thus, the problem with the exact model of f(x) is solved using the approximation of f(x). To approximate f(x) the same method can be applied but instead of considering the integral of the vector $\overrightarrow{T_x}$ as the expression for f(x), the expression for f(x) can be approximated by summation of the change in y value (the height) of each segment of the wire. This is expressed mathematically as:

$$f = \sum_{i=1}^{n} \Delta y_i$$

Where Δy is the segment's height, and n is the total number of segments. To find Δy the properties of the vector $\overrightarrow{T_c}$ can be applied. The vector $\overrightarrow{T_c}$ is the same as the vector $\overrightarrow{T_x}$ but it is the tension vector tangential to segment number c rather than point x. Thus $\overrightarrow{T_c}$ is:

$$\overrightarrow{T_c} = -\begin{pmatrix} \vec{R} \\ \vec{p} + \overrightarrow{F_B} + \overrightarrow{F_g} \end{pmatrix}$$

If each segment of wire is straight and considered to be 1 unit long, since tension vectors in a wire are always tangential to the wire; the tension vector at segment c is colinear with the segment number c. Thus the height of any segment c is equal to the y component of the unit vector in the direction of $\overrightarrow{T_c}$, which is mathematically expressed as:

$$\Delta y_c = \left|\frac{\overrightarrow{T_c}}{|\overrightarrow{T_c}|}\right| \sin\alpha$$

since:

$$\left|\frac{\overrightarrow{T_c}}{\left|\overrightarrow{T_c}\right|}\right| = 1$$

$$\Delta y_c = \sin\alpha$$

Where α is the angle between the vector $\overrightarrow{T_c}$ and the x axis. Thus, for a wire with one-unit long straight segments

$$f = \sum_{i=1}^{n} \sin\alpha_i$$

Where angle α is equal to the angle formed by the intersection of the vector $\overrightarrow{T_c}$ and a place holder vector parallel to the x axis $\vec{X}$:

$$\alpha = cos^{-1}\left(\frac{\vec{X}\cdot\overrightarrow{T_c}}{\left|\vec{X}\right|\left|\overrightarrow{T_c}\right|}\right)$$

Since $\vec{X}$ is parallel to the x axis:

$$\vec{X} = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\alpha = cos^{-1}\left(\frac{x\ component\ of\ \overrightarrow{T_c}}{\left|\overrightarrow{T_c}\right|}\right)$$

Thus:

$$f = \sum_{i=1}^{n} \sin\left(cos^{-1}\left(\frac{x\ component\ of\ \overrightarrow{T_c}}{\left|\overrightarrow{T_c}\right|}\right)\right)_i$$

Which can be simplified to:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{x\ component\ of\ \overrightarrow{T_c}}{\left|\overrightarrow{T_c}\right|}\right)^2}_i$$

by use of the following trigonometric simplification:

$$sin^2 x + cos^2 x = 1$$

Thus:

$$sinx = \sqrt{1 - cos^2 x}$$

Hence:

$$\sin(cos^{-1}x) = \sqrt{1 - cos^2(cos^{-1}x)} = \sqrt{1 - x^2}$$

Thus, the function that approximates the depth of the towfish tethered to a real wire in its fully expanded form is:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{\frac{1}{2}\rho_{water} C_D A \vec{v}^2}{\sqrt{\left(\frac{1}{2}\rho_{water} C_D A \vec{v}^2\right)^2 + (\vec{g}(\Delta\rho i V_n + \rho_{water} V - m))^2}} \right)^2}$$

The function f presented above only estimates the depth of the towfish tethered to a real wire, when the wire is split into n number of segments of length 1m. The function f is limited to estimating the depth of the towfish when the wire is split into 1-meter long segments because the unit vector of $\overrightarrow{T_c}$ was used to derive it.

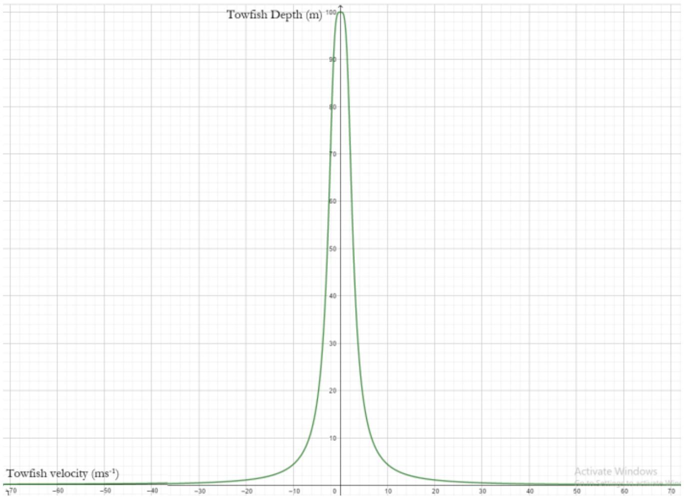

Figure 8 Plot of function f(v) showing the depth of the towfish when tethered to a real wire plotted against its velocity

In figure 8 each of the parameters have been given a realistic value (for the investigation it's not of importance to mention the value) and the argument of the function has been set as the velocity of the towfish $\vec{v}$. Figure 8 shows that the function f just like D (the function for the depth of the towfish tethered to an ideal wire) takes the shape of a bell function rapidly converging to 0, and just like D it is supported by the fact that the maximum depth the towfish can reach is the same as the length of the wire. It can also be seen in figure 8 that the function D and f behave similarly as the velocity of the towfish is varied, which further supports the validity of f and D since they both model the same thing, but with different simplifications and assumptions, this pattern is further highlighted in figure 9.

Figure 9 shows that the function f($\vec{v}$) and D($\vec{v}$) are almost identical when the real wire is very thin (0.001 mm), which is consistent with the assumptions used to derive the functions. The function D($\vec{v}$) models the towfish depth with a neutrally buoyant straight wire, and when the real wire is very thin, the effect of the distributed force of gravity acting on the wire will be very low since the distributed force of gravity is expressed by:

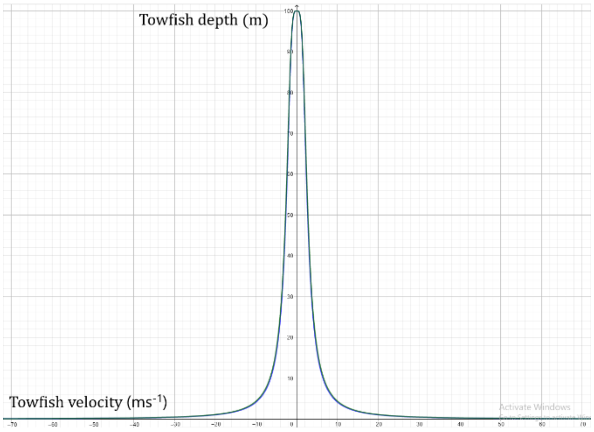

Figure 9 plot of function f(v) with a wire of radius 0.001 m and D(v)

$$\vec{p} = \vec{g}(\Delta\rho i V_n)$$

And when the radius of the wire is small V_n will be small thus the effect of the distributed force of gravity will be small. Thus when the radius of f($\vec{v}$) is very low d($\vec{v}$) is a good approximation of f($\vec{v}$). However as seen in figure 10 when the radius of the wire considered in f($\vec{v}$) is increased the discrepancy between the f($\vec{v}$) and D($\vec{v}$) is greatly increased, yet the behavior of the function is not significantly changed. Since the only difference between D($\vec{v}$) and f($\vec{v}$) is the consideration of $\vec{p}$, it can interestingly enough be concluded that f($\vec{v}$) is simply a translation of D($\vec{v}$). another interesting observation from figure 10 is that when the radius of the wire goes beyond 0.01 m the rate at which the function converges to 0 decreases significantly. Which indicates that varying the velocity of the towfish for large wires is not a valid system of depth control as very large velocity would be needed.

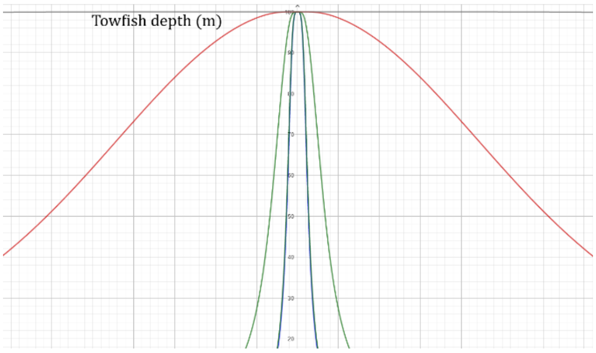

Figure 10 Plot of the function D(v) in blue and f(v) with different wire radii (Radious 0.01m: Green line, Radious 0.1m: Red line, Radious 1m: Grey line)

Behavior of the function as the geometry of the towfish is varied

Previously in the investigation the velocity of the towfish has been set as the argument of the function, and its behavior has been analyzed however varying the geometry of a towfish is another parameter which is vital to explore, as it too has a large effect on the depth of the towfish. To investigate the behavior of a towfish as the geometry is varied could be done for any towfish shape, however this exploration only concerns itself with the teardrop towfish design, as shown in figure 11, as it's the most modern design (Wikmark). The geometry of the towfish has a large impact on the drag acting on the towfish as well as the buoyant force acting on the towfish. Changing the geometry of the towfish has a large impact on the drag acting on the towfish because one of the parameters of drag is A (the cross sectional area of the towfish). Changing the geometry also impacts the buoyant force because one of the parameters of the buoyant force is V (volume of the towfish).

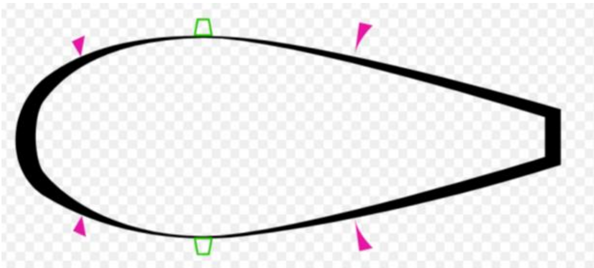

Figure 11 Image showing teardrop shape towfish (Arnero)

Since changing the geometry of the towfish changes 2 different parameters of the function f, these 2 parameters mut be expressed in terms of 1 parameter. This one parameter can be the parameter r (radius of the towfish). But since the towfish considered in this investigation is teardrop shaped, which is an irregular shape both V and A cant be simply expressed in terms of r since there is no standard volume equation available for a teardrop shape. Given that the teardrop towfish is defined by a function y(x) where y is the radius of the towfish at any point x, the parameters A and V can be defined using the derivative and integral of y(x). The parameter A can be defined as:

$$A = \pi {y_{max}}^2$$

Where y_{max} is the maximum radius of any point in the towfish. And V can be defined as the volume of rotation of the function y(x):

$$V = \int_a^b \pi y^2 dx$$

y_{max} is the y value of the maximum point of the function y(x), which can be found using the first derivative test:

$$y_{max} = y\left(\frac{dy}{dx} = (0)\right)$$

Which can be expressed in terms of the inverse function of the derivative of y:

$$y_{max} = y\left(\frac{dx}{dy}(0)\right)$$

Where $\frac{dx}{dy}$ is the inverse function of the derivative of y. using the above expression for y, the function can f can be expressed using the function y that describes the shape of the towfish:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{\frac{1}{2}\rho_{water} C_D \pi \left(y\left(\frac{dx}{dy}(0)\right)\right)^2 \vec{v}^2}{\sqrt{\left(\frac{1}{2}\rho_{water} C_D \pi \left(y\left(\frac{dx}{dy}(0)\right)\right)^2 \vec{v}^2\right)^2 + \left(\vec{g}\left(\Delta\rho i V_n + \rho_{water} \int_a^b \pi y^2 dx - m\right)\right)^2}} \right)^2}$$

This version of the function shows how the radious of the towfish at any point x is related to the depth of the towfish. As an example plot of this function for a varrying towfish radious, the function was set as:

$$y(x) = \frac{1}{2}\sqrt{1 - x^2}\left(1 - \frac{x+1}{2}\right)$$

Which is a funtion showing how the radious of a teardrop shaped towfish varries across its body. To plot figure 12 showing how varrying the radious of the towfish changes its depth, the function y(x) was solved for A and V so the function returned to its original form:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{\frac{1}{2}\rho_{water} C_D x A \vec{v}^2}{\sqrt{\left(\frac{1}{2}\rho_{water} C_D x A \vec{v}^2\right)^2 + \left(\vec{g}\left(\Delta\rho i V_n + \rho_{water} x V - m\right)\right)^2}} \right)^2}$$

Where x represents a multiple of the radius of the towfish at any point.

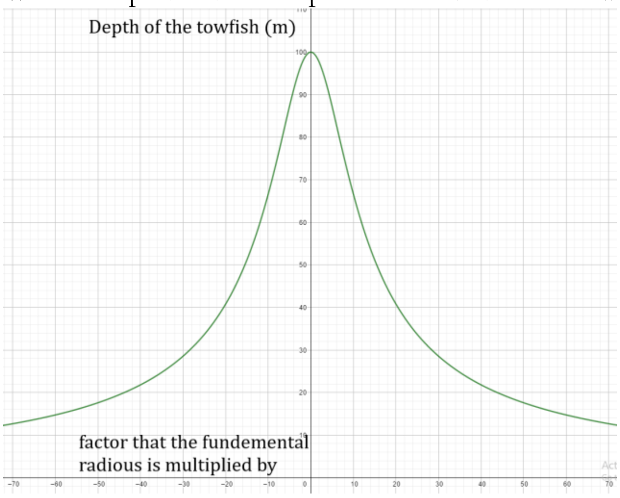

Figure 12 Plot of f(x) showing how multiplying the radius at any point of the towfish by x effects the depth of the towfish

In figure 12 all parameters other than x were set as constant. Interestingly figure 13 shows that varying the geometry of the towfish has a far smaller effect on the depth of the towfish than varying the velocity of the towfish. Also interestingly it shows that when varying the geometry of the towfish the function doesn't converge to 0 as it does when varying the velocity of the towfish, the limit of f(x) as x approaches infinity is 1.33, which shows that at some point varying the geometry of the towfish has no effect on the depth of the towfish.

<u>Conclusion</u>

The goal of this exploration was to investigate a function that models the depth of the towfish in response to several parameters. Throughout this investigation, trigonometry and vector techniques have been used to conjecture a function that models the depth of a towfish in response to varying its environmental conditions. Throughout this exploration 2 functions were found which model the depth of the towfish.

First a function that estimated its depth when assuming that the wire connected to the towfish is straight and rigid was found:

$$D = l\frac{-\vec{g}(V\rho - m)}{\sqrt{\left(\frac{1}{2}\rho C_D A\vec{v}^2\right)^2 + (\vec{g}(V\rho - m))^2}}$$

This function largely acted at as a stepping stone for further exploration however, interestingly it was later found that this function was a very close estimate to the function that models the depth of the towfish tethered to a non-rigid wire, which is significant as this function required far less computing power to plot than the function for the non rigid wire. It was also found that when increasing the wires radius past 0.01 m the discrepancy between the function modeling the rigid and non rigid wire became so vast that using it becomes inaccurate. To deal with this a function that models the towfish when tethered to an ideal wire was found too:

$$f = \sum_{i=1}^{n} \sqrt{1 - \left(\frac{\frac{1}{2}\rho_{water} C_D A\vec{v}^2}{\sqrt{\left(\frac{1}{2}\rho_{water} C_D A\vec{v}^2\right)^2 + (\vec{g}(\Delta\rho i V_n + \rho_{water} V - m))^2}}\right)^2}$$

However this function is only an estimate of the depth of the towfish when the wire tethered to the towfish is considered as a collection of 1 meter rigid wire segments attached together rather than a long continuous non rigid wire. This is the largest limitation of the function, its unknown how much the error of this function is because there is no empirical data available, however the consulting engineer (Frank Wikmark of marex consult) confirmed that the behavior of the functions derived in this investigation is consistent with his experience.

Although this exploration isn't fully comprehensive it provides a base function for future investigations into advanced towfish techniques. Future explorations into this topic are encouraged to explore how turbulence and complex water currents impact the behavior of the functions found in this exploration. Although this exploration wasn't as vast as originally anticipated, it managed to explore the basic concepts and behaviors of a function modeling the depth of a towfish.

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